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M.Sc. (Sem. - III) Examination October - 2025

Mathematics (Paper: PGMTH - 3044) Diophantine Equations

Time: 3 Hours] **[Total Marks: 70**

સૂચના : / Instructions	Seat No.:
(1) ઉપરોક્ત દર્શાવેલ વિગતો ઉત્તરવહી પર અવશ્ય લખવી. Fill up strictly the above details on your answer book	
(2) Follow usual notations and conventions.	
(3) Figures to the right indicate full marks of the corresponding question.	
(4) Use of non-programmable scientific calculator is permitted.	Student's Signature

Q. 1 **Answer any TWO of the following:** **(14)**

(1) For any finite simple continued fraction prove that the convergents with even / odd subscripts form a strictly increasing / decreasing sequence respectively. Also show that every convergent with an odd subscript is greater than every convergent with an even subscript.

(2) For finite simple continued fraction $[a_0; a_1, a_2, ..., a_n]$, if $C_k = \frac{p_k}{q_k}$ is the k^{th} convergent then prove that $p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1}$; $1 \le k \le n$. Use it to find the *gcd* (p_k, q_k) for all k.

(3) (a) Explain the method of solving linear Diophantine equation $ax + by = 1$ using continued fractions.

(b) From any two consecutive convergents to an irrational number x, prove that at least one, say $\frac{a}{b}$ satisfies $\left|x - \frac{a}{b}\right| < \frac{1}{2b^2}$.

Q. 2 **Answer any TWO of the following:** **(14)**

(1) If u, v is the fundamental solution and x_n, y_n gives all the solutions of $x^2 - dy^2 = 1$, then prove that $x_{n+1} = 2ux_n - x_{n-1}$, $y_{n+1} = 2uy_n - y_{n-1}$; for $n \ge 2$. Use it to obtain recurrence relation for x_n and y_n for $x^2 - 50y^2 = 1$.

(2) Prove that a linear Diophantine equation $ax + by = c$ has a solution if and only if gcd (a, b) divides c. Also obtain general formula for all its solutions, if (x_0, y_0) is any particular solution of this equation.

(3) Dev is purchasing beverages for a party. He has Rs. 68 to spend. He can purchase regular bottles of cold drink for Rs. 12, or smaller bottles for Rs. 8.00. How many ways are there to purchase beverages if he spends all of his money?

Q. 3 Prove that all the solutions of an equation $x^2 + y^2 = z^2$ satisfying the conditions $x > 0, y > 0, z > 0, 2 \mid x$ and gcd $(x, y, z) = 1$ are given by $x = 2st$, $y = s^2 - t^2$, $z = s^2 + t^2$; where s and t are positive integers such that $s > t$, gcd$(s, t) = 1$, $s \not\equiv t$ (*mod* 2) and conversely.
Also, for such primitive Pythagorean triple, prove that $12 \mid xy$ **(14)**

OR

Q. 3 (1) Prove that the radius of inscribed circle of a Pythagorean triangle is always an integer. **(07)**

(2) (a) If Fermat's last theorem is true when $n = 4$ or an odd prime, then prove that it is true for all $n > 2$.

(b) If the Diophantine equation $x^4 - y^4 = z^2$ has a solution, then prove that $gcd(x, y) = 1$. **(07)**

Q. 4 (1) Prove that an odd prime p can be expressed as a sum of two squares if and only if p is not of the form $4k + 3$. **(07)**

(2) Show that there are infinitely many positive integers which cannot be expressed as sums of three squares. Hence obtain all such integers between 25 and 35. **(07)**

OR

Q. 4 Prove that any prime p can be written as the sum of four squares. Hence prove that every natural number n can be written as the sum of four squares, some of which may be zero. **(14)**

Q. 5 **Answer any TWO of the following:** **(14)**

(1) If x, y, z is a primitive Pythagorean triple, then show that one of the integers x or y is even, while the other is odd. Also show that these integers are pair wise relatively prime.

(2) Prove that the k[th] convergent of the simple continued fraction $[a_0; a_1, a_2, ..., a_n]$ has the value $C_k = \frac{P_k}{q_k}$; $0 \le k \le n$, and obtain the recurrence relation for the p_k values of p_k and q_k .

(3) If x, y, z is a primitive Pythagorean triple, then show that one of the integers x or y is even, while the other is odd. Also show that these integers are pair wise relatively prime.